Journal of Statistical Physics, Vol. 98, Nos. 5/6, 2000

On the Brazil Nuts Problem: Is It of Relevance to Hydrodynamics?

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Received April 28, 1999; final October 18, 1999

A recent advance in understanding the "Brazil nuts phenomenon" together with insight into the relative motion of different-size particles in a lattice fluid makes this phenomenon amenable to an approach based on hydrodynamic concept. Here we use this conjecture as a stimulus for solving the equation describing asymmetric transport of particles in a medium moving with constant velocity. We also consider a method for constructing invariant solutions, which enables us to study the dependence of the probability density function on the parameters controlling the asymmetry of the flow. Physical realizations of the invariant solutions are in agreement with recent computer simulations by Alexander and Lebowitz.

KEY WORDS: Size segregation; driven lattice gas model; invariant and symmetries of partial differential equation.

I. INTRODUCTION AND APPROACH OF STUDY

Let us first provide an intuitive picture of the basic phenomenon. When a can containing one large ball and a number of smaller ones is shaken, the large ball rise to the top, even when the larger ball is more dense than the others or when the size ratio is not far from one. The Brazil nuts phenomenon refers to the manner in which transportable quantities are carried as a result of vibratory motion. This phenomenon is the canonical example of size segregation of particulate matter produced by shaking and it is important dynamical process common to many industrial systems and involves many physical processes. This dynamical process offers some fascinating theoretical difficulties. First, it is an example of simple, mechanical system displaying non-equilibrium and counterintuitive

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0022-4715/00/0300-1395$18.00/0

behaviour. Second, its understanding calls for a clear view of the dynamics and statistics of the size segregation processes. The Brazil nuts problem with a set of equivalent problems including interface motion, directed paths or polymers in random media and driven diffusion has recently reached a paradigmatic status in non-equilibrium statistical physics.[3–6] Because of its practical importance, particle-size segregation has been studied extensively during the past few decades with different approaches.[7] One of the fundamental contributions to the understanding of this phenomenon was the work of RSPS,[8, 9] where an adaption of the Monte Carlo method, commonly used in statistical mechanics, was used to study the dynamics of size segregation. Their computer simulations have several conclusions and detailed information concerning the dependence of segregation rate on size ratio and the scaling of segregation behaviour with the particle sizes and the distance through which they are lifted and their results indicate that lateral shaking has relatively ineffective in promoting segregation. Information obtained from these simulations was useful in developing an analytical description of the size segregation process carried out by AL,[1, 2] it may be a major contribution in this area. Where a good chance for developing a satisfactory theoretical understanding of this dynamical process exists in the context of fully developed series of computer simulations of non-equilibrium multicomponent lattice gases and polymers in solution, which exhibiting such segregation. Namely, they modeled the segregation of different sized particles as a gas of monomers and a single rod on a two-dimensional square lattice. A monomer occupies one site and the rod more than one. The particles interact by hard-core exclusion; no more than one particle of each type is permitted per site and jump to the neighbouring unoccupied lattice site, which leads to different types of diffusive flux and advective or drift. They studied both the diffusive and driven versions of this model, which is a modification of the basic driven lattice gas, known in the mathematical literature as the asymmetric simple exclusion process,[10, 11] and they also considered the case of a fixed rod corresponding to flow around an obstacle. Indeed, description of the flow field as a function of space-time to be an extremely challenging area of research. Frequently, to make the models tractable, the problem is reduced by certain forms of discretized and continuum analogue in limit. Thus a strong need exists to accurately model the particles motion. Alexander and Lebowitz[1] succeeded in describing some related continuum models whose behaviour is quite similar to that of polymermonomer lattice models which have relevance for the relative motion of different size particles induced by shaking. Their computer simulations presented an interesting and unexpected dependence of the density in front of the rod as a function of its length. The net results of their simulations show depend on the density profiles of the particles as seen from the large moving particle and depends also on the details of the jump rate. As succinctly noted by AL[1] "the effect seems to have a hydrodynamic explanation. However, we have not shown that this is the appropriate hydrodynamic limit of our model. This remains an open problem." Although, the computer simulation is viable alternative, it has the disadvantage of not providing an analytical expression of the stream function. In the main part of our recent paper,[12] and of the present work we intend to give an answer to this thesis by presenting explicit analytic solutions of the governing equations of the models suggested in ref. 1. Recently, we have discussed in detail properties and application of one of these models, namely; the driven diffusive flow past an impenetrable obstacle, where the nonlinear drift term assumed to be perpendicular to the direction of the diffusion term. The exact analytic solutions of the governing equation of this model presented a quantitative and qualitative understanding of the density profile of the particles on the front and back side of the moving obstacle. These exact analytical solutions, which are explicit and accurate, have allowed us to examine the sensitivity of the model to several important parameters.

In this paper, we continue our investigation by considering another model of the governing equation of the fluid in laminar (or turbulent) flow, where the diffusion term assumed to be in two-directions and drift term in one direction. To our knowledge, a detailed analysis leads to exact analytic solution has not been performed for this model, and therefor desirable. In the this respect the present work complements ref. AL.[1] The plan of the paper is as follows: in Section II we describe the mathematics formulation of the model. Section III is entirely devoted to show how the powerful method of Lie groups can be used to generate invariant solutions of the governing equation, which is an elliptic partial differential equation in $(1+2)$-dimension, in the way to be described shortly. Section IV contains conclusions and physical realization of these invariant solutions is discussed.

II. DESCRIPTION AND MATHEMATICAL FORMULATION OF THE PROBLEM

To provide a clear view of the local and geometric mechanism by which the size segregation is produced, it is necessary to know the density of the particles around the obstacle surface. We follow AL[1] to model the shaking process exhibiting the dynamical behaviour of two component mixture by which the non-equilibrium state with the large particles on the top. For this we consider a driven diffusive flow of particles past rigidly aligned obstacle, for simplicity, we shall take the obstacle to be a strip of width L, parallel to z-axis, and normal to the incident flow. The flux of the particles in a laminar (or turbulent) flow, which has diffusive term $-D(u_x i + u_y j)$, where $u = u(x, y, t)$ is a function describing, within significant time-space scale, the concentration of the particles at the moment t, and (x, y) are dimensionless spatial coordinates and D is the constant diffusion coefficient, and i, j are unit positive directions. If the motion of the particles is impeded by the presence of the strip (large) particle, so the drift term, which may result from the motion of the fluid, will arise. Let us consider the case where the drift is proportional to the width of the strip L and the constant velocity of the fluid v, which we take it to be in the i-direction and is oriented along the particle motion, thus the drift is $vLui$ and the total current is

$$J(x, y, t) = -D(u_x i + u_y j) + vLui \tag{1}$$

Equation of conservation of particles in the system now appears as

$$u_t = D(u_{xx} + u_{yy}) - vLu_x \tag{2}$$

where subscript represents partial derivative. In the steady state, Eq. (2) reads

$$(vL/D)\, u_x = (u_{xx} + u_{yy}) \tag{3}$$

If we consider the origin of the moving frame of the strip at the center of the strip, then strip intersect the xy-plane along the interval $y = L/2$ to $-L/2$ on the y-axis. The half-width L may be used as the unit length, then the strip occupies the interval $-1 \leqslant y \leqslant 1$ on y-axis. In addition, we require that $u(x, y, t)$ be non-negative and finite at the regions of the (x, y)-plane juste beside the strip for the long time, in the front and back side of the obstacle. In the following we shall solve this problem exactly for (vL/D) large and also for small values, which represents the density profile of the particles around the obstacle surface. In the course for doing that, we will be utilizing the Lie group analysis which exploits the symmetries of Eq. (2) and (3) to derive some ansatz leading to reduction of the variables, where the analytic solutions are easier to obtain and an elegant closed form solutions exists.

III. EXPLICIT INVARIANT SOLUTIONS

The basic concepts and equations of Lie groups were introduced in some detail on numerous previous occasions by the author, e.g., ref. 13 and references therein, so that we may confine ourselves to a short introduction coupled with a summary of the main equations. We search for infinitesimal-Lie point transformations of the form

$$\begin{aligned}\bar{x} &= x + \varepsilon V^x(x, y, t, u) + o(\varepsilon^2)\\ \bar{y} &= y + \varepsilon V^y(x, y, t, u) + o(\varepsilon^2)\\ \bar{t} &= t + \varepsilon V^t(x, y, t, u) + o(\varepsilon^2)\\ \bar{u} &= u + \varepsilon V^u(x, y, t, u) + o(\varepsilon^2)\end{aligned} \tag{4}$$

Which leave the Eq. (2) invariant to $o(\varepsilon)$; that is to say, that, to $o(\varepsilon)$,

$$\bar{u}_{\bar{t}} = D(\bar{u}_{\bar{x}\bar{x}} + \bar{u}_{\bar{y}\bar{y}}) - vL\bar{u}_{\bar{x}} \tag{5}$$

wherever Eq. (2) hold. Using the differential forms approach of Harrison and Estabrook,[14] the invariance of Eq. (5) to $o(\varepsilon)$ yields a set of determining equations for the isovector field $V = (V^x, V^y, V^t, V^u)$.

The general isovector

$$V = V^x \frac{\partial}{\partial x} + V^y \frac{\partial}{\partial y} + V^t \frac{\partial}{\partial t} + V^u \frac{\partial}{\partial u} \tag{6}$$

of the Lie algebra can be decomposed in terms of basis vectors $X_1, X_2, \ldots, X_9$ as

$$V = I_1 X_1 + I_2 X_2 + \cdots + I_9 X_9 \tag{7}$$

where $I_1, I_2, \ldots, I_9$ are arbitrary constants, and

$$\begin{gathered} X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial t}, \qquad X_4 = \frac{1}{2D} u \frac{\partial}{\partial u}\\ X_5 = t\frac{\partial}{\partial y} + \frac{1}{2D} u[-y] \frac{\partial}{\partial u}, \qquad X_6 = t\frac{\partial}{\partial x} + \frac{1}{2D} u[-x + vLt] \frac{\partial}{\partial u}\\ X_7 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + \frac{1}{2D} u[vLy] \frac{\partial}{\partial u},\\ X_8 = t\frac{\partial}{\partial t} + \frac{1}{2}x\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial y} + \frac{1}{2D} u\left[\frac{vL}{2}x - \frac{v^2L^2}{2}t\right]\frac{\partial}{\partial u}\\ X_9 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y}\\ + \frac{1}{2D} u\left[-\frac{1}{2}x^2 - \frac{1}{2}y^2 + vLxt - 2Dt - \frac{v^2L^2}{2}\right]\frac{\partial}{\partial u}\end{gathered} \tag{8}$$

The group invariant are functions $H(\bar{x}, \bar{y}, \bar{t}, \bar{u}) = H(x, y, t, u)$, where Taylor expansion for small ε yields the infinitesimal invariance condition

$$\left[V^x \frac{\partial}{\partial x} + V^y \frac{\partial}{\partial y} + V^t \frac{\partial}{\partial t} + V^u \frac{\partial}{\partial u} \right] H(x, y, t, u) = 0 \tag{9}$$

The group trajectories

$$\frac{dx}{V^x} = \frac{dy}{V^y} = \frac{dt}{V^t} = \frac{du}{V^u} = d\varepsilon \tag{10}$$

may be integrated to determine the functionally independent invariant $s(x, y, t)$, $z(x, y, t)$ and dependent invariant $K(s, z)$. The reductions are constructed by imposing these invariant into Eq. (2) to get some partial differential equations of two variables $K(s, z)$ only. To get the reduced ordinary differential equations, we have to apply once more this procedure to the resultant equations. Solutions of these ordinary differential equations lead by back substitution to so-called invariant solutions $u(x, y, t)$ of Eq. (2). These solutions in general form a large class of solutions that possess a type of invariance under certain transformations (stretching, rotations, translations, etc.) of the variables. This means that the motion of the system having time t and spatial (x, y) as independent variables is one in which the dependent variable (u) or parameters that characterize the system vary in such a way that, as time evolves, the spatial variation of these parameters remains geometrically similar. Let us now consider some types of these exact solutions of Eq. (2):

Case 1. The system of invariant which corresponds to the operator X_7 in Eq. (8), is

$$s = \frac{x^2 + y^2}{2}, \qquad z = t \tag{11}$$

therefore, an invariant solution can be found in the form

$$u(x, y, t) = \exp\left(\frac{vL}{2D} x\right) \qquad K(z, s) \tag{12}$$

where the function K satisfies partial differential equation

$$K_z = 2D(sK_{ss} + K_s) - \frac{v^2 L^2}{4D} K \tag{13}$$

Equation 13 can be transformed into ordinary differential equation (ODE) by using the ansatz $K(s, z) = \exp(-wz) \cdot h(s)$, where $h(s)$ satisfies

$$s\frac{d^2h}{ds^2} + \frac{dh}{ds} - \left(\frac{w}{2D} + \frac{v^2L^2}{8D^2}\right)h = 0 \tag{14}$$

and w is arbitrary constant. Equation (14) has the general solution in the form of linear combination of cylindrical functions

$$h(s) = c_1 I_0(b\sqrt{s}) + c_2 K_0(b\sqrt{s}), \qquad b^2 = \frac{-4wD + v^2L^2}{2D^2} \tag{15}$$

where I_0 and K_0 are modified Bessel functions. Thus the invariant solution of Eq. (2) can be written on the form

$$u(x, y, t) = \exp\left(\frac{vL}{2D}x - wt\right)\left[c_1 I_0\left(b\sqrt{\frac{x^2+y^2}{2}}\right) + c_2 K_0\left(b\sqrt{\frac{x^2+y^2}{2}}\right)\right] \tag{16}$$

The corresponding stationary solution of Eq. (3) can be derived from Eq. (16) by letting arbitrary constant $w = 0$,

$$u(x, y) = \exp\left(\frac{vL}{2D}x\right)\left[c_1 I_0\left(a\sqrt{\frac{x^2+y^2}{2}}\right) + c_2 K_0\left(a\sqrt{\frac{x^2+y^2}{2}}\right)\right] \tag{17}$$

where $a^2 = v^2L^2/2D^2$.

Case 2. Equation (2) admits the operator $X_2 + wX_3$, then invariant solution corresponding to this operator is to be found in the form $s = y - wt$, $z = x$ and $u(x, y, t) = K(s, z)$, and from Eq. (2) we have the following equation for K,

$$D(K_{ss} + K_{zz}) - vLK_z + wK_s = 0 \tag{18}$$

For $K(s, z) = \exp(vLz/2D)\,h(s)$, Eq. (18) can be transformed into ODE,

$$\frac{d^2h}{ds^2} + \frac{w}{D}\frac{dh}{ds} - \left(\frac{v^2L^2}{4D^2}\right)h = 0 \tag{19}$$

which has the exact solution

$$h(s) = Ae^{cs} + Be^{rs}$$

where $c=(-w+\sqrt{w^2+v^2L^2})/2D$, $r=(-w-\sqrt{w^2+v^2L^2})/2D$ and A, B are arbitrary constants. Turning back to the former variables, we come to the following formula

$$u(x, y, t)=\exp((vLx-wy+w^2t)/2D)[A\cosh a(y-wt)+B\sinh a(y-wt)]$$
$$\text{where}\quad a^2=\sqrt{(w^2+v^2L^2)/4D^2} \tag{21}$$

The exact solution of Eq. (3), for the stationary case, is to be found using Eq. (21) by $w=0$

$$u(x, y)=\exp\left(\frac{vL}{2D}x\right)\left[A\cosh\left(\frac{vL}{2D}y\right)+B\sinh\left(\frac{vL}{2D}y\right)\right] \tag{22}$$

Case 3. Invariant solution corresponding to linear combination of the operators $vLX_1+X_2+X_3+X_4$ is to be found in the form

$$s=x-vLt,\qquad z=y-t\qquad\text{and}\qquad u(x, y, t)=\exp(t/2D)\,K(z, s)$$

where $K(s, z)$ satisfies

$$D(K_{ss}+K_{zz})+K_z-\frac{1}{2D}K=0 \tag{23}$$

Now, to reduce Eq. (23) to ODE, symmetry reduction method tells that, for $K(s, z)=\exp(z/2D)\cdot h(s)$, Eq. (23) reduced to

$$\frac{d^2h}{ds^2}+\frac{1}{4D^2}h=0 \tag{24}$$

which has the general solution

$$h(s)=A\cos\left(\frac{s}{2D}\right)+B\sin\left(\frac{s}{2D}\right) \tag{25}$$

where A and B are constants.

Solution $h(s)$ leads by back substitution to general solution of Eq. (2), as

$$u(x, y, t)=\exp\left(\frac{y}{2D}\right)\cdot\left[A\cos\left(\frac{x-vLt}{2D}\right)+B\sin\left(\frac{x-vLt}{2D}\right)\right] \tag{26}$$

Case 4. Operator X_5 suggests the following ansatz,

$$s = t \quad \text{and} \quad z = x \quad \text{and} \quad u(x, y, t) = \exp(-y^2/4Dt) \cdot K(s, z)$$

where $K(s, z)$ satisfies

$$DK_{zz} - vLK_z - K_s - \frac{1}{2s} K = 0 \tag{27}$$

Equation (27) can be reduced to ODE

$$\frac{dh}{ds} + \frac{4D + v^2L^2s}{4D} h = 0 \tag{28}$$

where

$$K(s, z) = \exp\left(\frac{vLz}{2D} - \frac{z^2}{4Ds}\right) h(s)$$

Equation (28) has the solution

$$h(s) = As^{-1} \exp\left(\frac{-v^2L^2s}{4D}\right), \qquad A \text{ is arbitrary constant}$$

Turning back to the former variables, we come to the following solution

$$u(x, y, t) = \frac{A}{t} \exp\left(-\frac{x^2 + y^2}{4Dt} + \frac{vLx}{2D} - \frac{v^2L^2t}{4D}\right) \tag{29}$$

On the other hand, the partial differential equation (PDE) (27) can be reduced to the following ODE

$$\frac{d^2h}{dz^2} - \frac{vL}{2D}\frac{dh}{dz} = 0 \tag{30}$$

where $K(s, z) = s^{-1/2} \cdot h(z)$.

Equation (30) has the solution

$$h(z) = A + B \exp\left(\frac{vL}{D} z\right), \qquad A \text{ and } B \text{ are arbitrary constants}$$

Thus, we have solution of Eq. (2) in the form

$$u(x, y, t) = \frac{1}{\sqrt{t}} \exp\left(\frac{-y^2}{4Dt}\right)\left[A + B \exp\left(\frac{vL}{D} x\right)\right] \tag{31}$$

Case 5. In addition, within the process of symmetry reduction by the operator X_6, we get the following ansatz;

$$s = t, \qquad z = y \qquad \text{and} \qquad u(x, y, t) = \exp\left(\frac{-x^2}{4Dt} + \frac{vLx}{2D}\right) \cdot K(s, z)$$

and $K(s, z)$ satisfies the PDE

$$K_s = DK_{zz} - \frac{2D - v^2L^2s}{4Ds} K \tag{32}$$

which can be reduced to ODE

$$\frac{d^2h}{dz^2} = \frac{v^2L^2}{4D^2} h \tag{33}$$

where $K(s, z) = s^{-1/2}h(z)$.

Equation (33) has the solution

$$h(z) = \left[A \cosh\left(\frac{vL}{2D} z\right) + B \sinh\left(\frac{vL}{2D} z\right)\right]$$

Solution $h(s)$ leads by back substitution to the following solution of Eq. (2)

$$u(x, y, t) = \frac{1}{\sqrt{t}} \exp\left(\frac{-x^2}{4Dt} + \frac{vLx}{2D}\right)\left[A \cosh\left(\frac{vL}{2D} y\right) + B \sinh\left(\frac{vL}{2D} y\right)\right] \tag{34}$$

where A and B are arbitrary constants.

Case 6. Applying the symmetry reduction method to Eq. (3), for the stationary case, yields the ansatz, $z = x - y$ and $y(x, y) = h(z)$, where $h(z)$ satisfies

$$\frac{d^2h}{dz^2} - \left(\frac{vL}{2D}\right)\frac{dh}{dz} = 0 \tag{35}$$

which has the solution $h(z) = z \exp(vLz/2D)$. Thus, Eq. (3) has solution of the form

$$u(x, y) = A \exp\left(\frac{vL(x - y)}{2D}\right) \tag{36}$$

IV. PHYSICAL REALIZATION OF THE INVARIANT SOLUTIONS

In studying the Brazil nuts phenomenon, where the particles interact by hardcore exclusion, it is necessary to know the density profile around the obstacle surface during the shaking process. We have presented four different types of time-dependent solutions of Eq. (2), which has relevance for the size segregation process. Physical sense of this expression is in its describing the concentration around the moving obstacle, where depend on time means depend on repeated shaking process. The common feature of these solutions is that, for long time the concentration $u(x, y, t)$ around the obstacle is bounded and decrease monotonically to reach $u = 0$ as $t \to \infty$, and consequently the large particle will move in this direction. In analyzing the spatial-depend of the concentration, one has to keep in view that the regions under consideration are just up the obstacle and down it, i.e., at $x = 0^-$ and $x = 0^+$, and $-1 \leqslant y \leqslant 1$. According as the geometric of the problem, the physical solution requires symmetry of y-axis. We shall show that this is indeed the case, where $u(x, y) = y(x, -y)$, but $u(-x, y) \leqslant u(x, y)$, which indicates that the density will be non-uniform, and there is a difference in the concentration of the particles around the obstacle surface, which enables the large particle to move (respectively fast) up. On the other hand, it is of interest to note that the explicit dependence of these solutions on the parameters vL/D will illustrate the connection between them and the concentrations around different obstacle's length L and nature of the medium (v, D). For this, we have to distinguish two cases; $(vL/D) \gg 1$ and $(vL/D) \ll 1$. In order to avoid non-physical solutions, we have to choose the arbitrary constants in suitable manner. Let us now comment each type of these solutions in the light of this introduction.

1. The Modified Bessel Solution

Taking into account the properties of the modified Bessel functions; $I_0(c) \approx 1$ as $c \to 0$, but $K_0(c) \to \infty$ as $c \to 0$. Physical solutions can be fulfilled if we put the arbitrary constant $c_2 = 0$ in Eq. (16) for the case $(vL/2D) \ll 1$. But, for $(vL/2D) \gg 1$, we put $c_1 = 0$, to get the following

$$u(vL/2D) = \exp(vL/2D) \cdot I_0(vL/2D), \qquad (vL/2D) \ll 1$$

$$u(vL/2D) = \exp(vL/2D) \cdot K_0(vL/2D), \qquad (vL/2D) \gg 1$$

which indicates that the concentration of the particles has inverse proportional with L and v, i.e., in above of the obstacle, the density decrease monotonically with L and v, but the accumulation of the particles increase down the larger particle, and therefore large particles move up relative to the small particle.

2. The Exponential Type Solution

The time-dependent solutions in Eqs. (29), (31) of Eq. (2) present a possibility to discuss and understand why the large particles on top. At first glance, $u(x, y, t)$ is bounded as $t \to \infty$, and $u(-y) = u(y)$, $u(-x) < u(x)$, that is to say, the presence of an obstacle, in this model of driven diffusive flow, will distort the local concentration profile to a state which divided the (x, y)-plane into two regions, the concentration is relatively higher in one side than the other side apart from the parameter value of $vL/2D$. The effect of the parameters $vL/2D$ on the concentration profile may be interpreted as a catalytic or inhibitory agent to the motion of the obstacle up toward the top surface.

3. The Hyperbolic-Function Solution

The arbitrary constants in Eqs. (22), (34) can be chosen so that the density simulates the desired physical requirement $u(-y) = u(y)$ by letting $B = 0$. Although, the exponential function and hyper-cosine are monotonic increasing functions, but in the regions we are interested, even for $(vL/2D) \geqslant 1$, the density function is non-negative and bounded, and satisfies $u(-x) < u(x)$. This feature is characteristic of all time-dependent solutions of the governing Eq. (2), so the previous interpretation still valid.

4. The Trigonometric-Type Solutions

Although, the formal solution (26), not satisfies the geometric condition $u(-y) = u(y)$, it is non-trivial solution of Eq. (2). A useful property of this solution is that, for the arbitrary constant $B = 0$, we have non-negative and bounded solution and moreover it satisfies $u(-x) < u(x)$. This solution contain term $(x - vLt)$, which indicate that the particles will accumulate in a wave-like feature, and create difference in concentration values around the obstacle surface.

In addition to these four time-dependent solutions of Eq. (2), we have derived three different types of the governing equation of the steady state, Eq. (3). The solutions (17), (22), are obtained as a special case of the time-dependent solutions (16), (21). The solution (17) may by compared with that obtained in ref. 15 by different approach; method of asymptotic analysis. They were considering the flow of groundwater around a cylindrical obstacle.

We conclude that, the complete concordance between these theoretical dependence and computer simulations conducted by Alexander and Lebowitz[1] are convincingly support their conjecture that; the effect of moving obstacle in a driven diffusive flow, which has relevance for the Brazil nuts phenomenon, have a hydrodynamic explanation and Eq. (2) is the appropriate hydrodynamic limit of the model consists of a gas of monomers and a single rod on a lattice. In general context, hydrodynamic model of the problem of vibrated containers of granular material was the subject of number of theoretical proposals, with the aim of constructing continuum models, for example; convection in a vibrating container of granular material can be modeled by the Navier–Stokes equation.[7]

ACKNOWLEDGMENT

I am grateful to a referee for comments.

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